Algebraic structures related to reflection equations



1992 J. Phys. A: Math. Gen. 25 5963

(http://iopscience.iop.org/0305-4470/25/22/022)

J. Phys. A: Math. Gen. 25 (1992) 5963–5975. Printed in the UK

# Algebraic structures related to reflection equations

P P Kulish†§ and E K Sklyanin‡§

† Yukawa Institute for Theoretical Physics (YITP), Kyoto University, Kyoto 606, Japan
‡ Research Institute for Mathematical Sciences (RIMS), Kyoto University, Kyoto 606, Japan

Received 22 May 1992

**Abstract.** Quadratic algebras related to the *reflection equations are introduced*. They are quantum group comodule algebras. The quantum group $F_q(\mathrm{GL}(2))$ is taken as the example. The properties of the algebras (centre, representations, realizations, real forms, fusion procedure etc) as well as the generalizations are discussed.

## 1. Introduction

Progress in understanding the algebraic roots of the quantum integrability achieved in the last few decades has already resulted in the introduction of several new algebraic objects, such as the Yang–Baxter equation (YBE), quantum groups and quantum algebras, exchange and quadratic algebras. It seems that the list should be enriched with another item: the reflection equation (RE), which has arisen recently in several independent contexts.

The RE reads

$$R(u-v)\overset{1}{K}(u)R(u+v)\overset{2}{K}(v)=\overset{2}{K}(v)R(u+v)\overset{1}{K}(u)R(u-v) \tag{1}$$

where $K(u)$ is a square $N\times N$ matrix, $\overset{1}{K}\equiv K\otimes \mathrm{id}_N$, $\overset{2}{K}\equiv \mathrm{id}_N\otimes K$ (the notation is usual in the quantum inverse scattering method), and $R(u)$ is a solution to the YBE. One can also consider (1) as defining relations for the associative algebra $\mathcal{A}$ generated by the elements of the matrix $K(u)$ [1].

The RE introduced in [2] as the equation describing factorized scattering on a half-line, and the related algebra $\mathcal{A}$ soon found quite different applications in quantum current algebras [3] and integrable models with non-periodic boundary conditions [1, 4].

Since the constant (i.e. not including the spectral parameter $u$) solutions to the YBE are extensively used in quantum group theory [5], it is quite natural to study a version of the RE without the spectral parameter. Though such an equation and the related algebras (for different $R$-matrices) has already appeared in several papers [6–12], they were not distinguished as separate objects of study until recently.

§ Permanent address: Steklov Mathematical Institute, Fontanka 27, St. Petersburg 191011, Russia.

In the present paper we collect the basic facts which are necessary for any systematic study of the RE. We restrict ourselves to the case of the simplest quantum group $F_q(\mathrm{GL}(2))$ and choose the following form of the RE

$$R \overset{1}{K} R^{t_1} \overset{2}{K} = \overset{2}{K} R^{t_1} \overset{1}{K} R. \tag{2}$$

This paper is organized as follows. After introducing the basic definitions and notation the general properties of the quadratic algebra $\mathcal{A}$ defined by (2) are described in section 2. They are discussed and partially proven in section 3. In section 4 a few comments are made on the representations of $\mathcal{A}$. In section 5 some equivalent variants of the RE are pointed out as well as the relations between $\mathcal{A}$ and other algebras. In conclusion, we discuss some generalizations of the RE and its applications.

## 2. Definitions

The quantum group $F \equiv F_q(\mathrm{GL}(2))$ can be defined as the associative algebra generated by four elements $a$, $b$, $c$, $d$ and the relations

$$\begin{array}{lll} ab = qba & bd = qdb & [a,d] = \omega bc \\ ac = qca & cd = qdc & [b,c] = 0 \end{array} \tag{3}$$

$q$ being a complex parameter and $\omega = q - q^{-1}$. Introducing the matrix $T$

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and using the notation $\overset{1}{T} \equiv T \otimes \mathrm{id}$, $\overset{2}{T} \equiv \mathrm{id} \otimes T$ one can rewrite the relations (3) in the compact form [5]

$$R \overset{1}{T} \overset{2}{T} = \overset{2}{T} \overset{1}{T} R \tag{4}$$

where the $R$-matrix is given by

$$R = \begin{pmatrix} q & & & \\ & 1 & & \\ & \omega & 1 & \\ & & & q \end{pmatrix} \qquad R^{t_1} = \begin{pmatrix} q & & & \omega \\ & 1 & & \\ & & 1 & \\ & & & q \end{pmatrix} \tag{5}$$

and $t_1$ means transposition with respect to the first space in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

*Remark.* If we replace the $R$-matrix in (4) by $\mathcal{P} R^{-1} \mathcal{P}$, where $\mathcal{P}$ is the permutation operator in $\mathbb{C}^2 \otimes \mathbb{C}^2$, the commutation relations (3) remain the same. The consequences of this fact are discussed in section 6.2.

The Hopf algebra structure is defined on $F$ after introducing the comultiplication map $\Delta : F \to F \otimes F$, co-unit map $\varepsilon : F \to \mathbb{C}$ and coinverse map $s : F \to F$ by the formulae

$$\Delta(T) = T_1 T_2 \qquad \varepsilon(T) = \mathrm{id} \qquad s(T) = T^{-1}$$

where $(T_1T_2)_{ij} \equiv \sum_{k=1}^{2} T_{ik} \otimes T_{kj}$.

Now define the associative algebra $\mathcal{A}$ by four generators $\alpha$, $\beta$, $\gamma$, $\delta$, which can be thought of as the elements of the matrix $K$

$$K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

and by the quadratic relations (2) or, more explicitly

$$\begin{aligned} &[\alpha,\beta] = \omega\alpha\gamma && \alpha\gamma = q^2\gamma\alpha && [\alpha,\delta] = \omega(q\beta+\gamma)\gamma \\ &[\beta,\gamma] = 0 && [\beta,\delta] = \omega\gamma\delta && \gamma\delta = q^2\delta\gamma . \end{aligned} \tag{6}$$

The algebra $\mathcal{A}$ has the following properties.

1. $\mathcal{A}$ is a Poincaré–Birkhoff–Witt (PBW) algebra which means that the linear space spanned by the monomials of order $p$ in the generators $\alpha\beta\gamma\delta$ has the same dimension as in the commutative case, that is $(p+3)!/p!3!$.

2. $\mathcal{A}$ is an $F$-comodule algebra that is there exists a map $\varphi$ (coaction of $F$) $\varphi : \mathcal{A} \to F \otimes \mathcal{A}$ which is consistent with the comultiplication $\Delta$

$$(\Delta \otimes \mathrm{id}) \circ \varphi = (\mathrm{id} \otimes \varphi) \circ \varphi \tag{7}$$

and the co-unit $\varepsilon$

$$(\varepsilon \otimes \mathrm{id}) \circ \varphi = \mathrm{id} \tag{8}$$

and, besides, is an algebra homomorphism.

By virtue of the duality between $F = F_q(\mathrm{GL}(2))$ and the quantum algebra $\mathcal{U}_q(\mathrm{sl}(2))$ the dual map $\varphi^* : \mathcal{U}_q(\mathrm{sl}(2)) \otimes \mathcal{A} \to \mathcal{A}$ defines the structure of $\mathcal{U}_q(\mathrm{sl}(2))$-module algebra on $\mathcal{A}$.

3. The centre of $\mathcal{A}$ is generated by two elements (for generic $q$, not root of unity)

$$c_1 = \beta - q\gamma \qquad c_2 = \alpha\delta - q^2\beta\gamma . \tag{9}$$

4. There are three real forms of $\mathcal{A}$ consistent with three known real forms of $F$: $F_q(\mathrm{U}(2))$, $F_q(\mathrm{U}(1,1))$ and $F_q(\mathrm{GL}(2,\mathbb{R}))$.

5. The two-sided ideal generated in $\mathcal{A}$ by the relation $c_1 = 0$ is invariant under the coaction $\varphi$ and the corresponding quotient algebra is isomorphic to the quantum homogeneous space.

## 3. Discussion and proofs

*3.1.* In order to prove the PBW property of $\mathcal{A}$ it is necessary to verify that any monomial in $\alpha\beta\gamma\delta$ can be expanded uniquely into the sum of alphabetically ordered monomials. The possibility of alphabetical ordering is easy to establish using the commutation relations (6). As for the linear independence of alphabetically ordered monomials, it is sufficient to verify it only for cubic monomials [13, 14].

*3.2.* The (left) coaction of $F$ on $\mathcal{A}$ is defined on the generators by the formula

$$\varphi(K) = TKT^t \tag{10}$$

where $t$ stands for the matrix transposition, and is extended to the whole algebra $\mathcal{A}$ as an algebra homomorphism. For example

$$\varphi(\beta) = ac\alpha + bc\gamma + ad\beta + bd\delta.$$

Verification of the $F$-comodule algebra axioms (7), (8) is a matter of direct calculation based on the commutation relation (4) and the equivalent relations

$$R\overset{1}{T}{}^t\overset{2}{T}{}^t = \overset{2}{T}{}^t\overset{1}{T}{}^t R \qquad \overset{1}{T}{}^t R^{t_1}\overset{2}{T} = \overset{2}{T} R^{t_1}\overset{1}{T}{}^t \qquad \overset{1}{T} R^{t_1}\overset{2}{T}{}^t = \overset{2}{T}{}^t R^{t_1}\overset{1}{T}$$

obtained from (4) using transpositions and the symmetries

$$\mathcal{P}R^t\mathcal{P} = R \qquad [\mathcal{P}, R^{t_1}] = 0 \tag{11}$$

of the $R$-matrix (5). Here superscript $t$ means total transposition and $\mathcal{P}$ is the permutation operator in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

The duality between the quantum group $F = F_q(\mathrm{GL}(2))$ and the quantum algebra $\mathcal{U}_q(\mathrm{sl}(2))$ [5, 15] implies that $\mathcal{A}$ is also a $\mathcal{U}_q(\mathrm{sl}(2))$-module algebra.

The algebra $\mathcal{U}_q(\mathrm{sl}(2))$ is generated by three elements $H$, $X_+$, $X_-$ and the relations

$$[H, X_\pm] = \pm X_\pm \qquad [X_+, X_-] = \frac{q^{2H} - q^{-2H}}{q - q^{-1}}$$

or in matrix form

$$R^\pm \overset{1}{L}{}^\pm \overset{2}{L}{}^\varepsilon = \overset{2}{L}{}^\varepsilon \overset{1}{L}{}^\pm R^\pm \qquad \forall\varepsilon \in \{+, -\}$$

where

$$L^+ = \begin{pmatrix} q^H & \omega X_- \\ 0 & q^{-H} \end{pmatrix} \qquad L^- = \begin{pmatrix} q^{-H} & 0 \\ -\omega X_+ & q^H \end{pmatrix} \tag{12}$$

and

$$R^+ = q^{-1/2}\mathcal{P}R\mathcal{P} \qquad R^- = q^{1/2}R^{-1}.$$

The pairing $\langle\, , \rangle$ between $F$ and $\mathcal{U}_q(\mathrm{sl}(2))$ is described by the relations [5]

$$\left\langle \overset{1}{L}{}^{\pm}, \overset{2}{T} \right\rangle = R^{\pm}_{12} \qquad \left\langle \overset{1}{L}{}^{\pm}, \overset{2}{T}\overset{3}{T} \right\rangle = R^{\pm}_{12}R^{\pm}_{13} \quad \ldots$$

where, as usual, the subscripts mark the spaces where the corresponding $R$-matrices act non-trivially.

The above formulae allow us to calculate the corresponding action $\varphi^*$ of $\mathcal{U}_q(\mathrm{sl}(2))$ on $\mathcal{A}$

$$\varphi^*(\overset{1}{L}{}^{\pm})\overset{2}{K} \to R^{\pm}_{12}\overset{2}{K}(R^{\pm}_{12})^{t_2}$$

or, more explicitly

$$\varphi^*(X_-)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ q^{-1}\alpha & \beta + q^{-1}\gamma \end{pmatrix}$$

$$\varphi^*(X_+)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} q\beta + \gamma & q\delta \\ \delta & 0 \end{pmatrix}$$

$$\varphi^*(q^H)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} q\alpha & \beta \\ \gamma & q^{-1}\delta \end{pmatrix}.$$

*3.3.* Before we proceed to the discussion of the centre of $\mathcal{A}$ let us recall few facts about the quantum group $F$. The quantum determinant $\det_q T$ generating the centre of $F$ can be constructed via the fusion procedure technique [5, 16]

$$P_-\overset{1}{T}\overset{2}{T} = P_-\overset{1}{T}\overset{2}{T}P_- = \det_q T P_-. \tag{13}$$

The rank-one projector $P_-$ and the complementary rank three projector $P_+$ in $\mathbb{C}^2 \otimes \mathbb{C}^2$ are defined by the spectral decomposition of the modified $R$-matrix [17]

$$\hat{R} \equiv \mathcal{P}R = qP_+ - q^{-1}P_-. \tag{14}$$

The commutativity of $\det_q T$ with $T$ and its group-like property $\Delta(\det_q T) = \det_q T \otimes \det_q T$ follow from the relation (4), e.g.

$$\Delta(\det_q T) = P_-\overset{1}{T}_1\overset{1}{T}_2\overset{2}{T}_1\overset{2}{T}_2 = P_-\overset{1}{T}_1\overset{2}{T}_1P_-\overset{1}{T}_2\overset{2}{T}_2 = \det_q T \otimes \det_q T.$$

It is well known that the quantum group $F_q(\mathrm{GL}(2))$ possesses the invariant quadratic form $\varepsilon_q$

$$\varepsilon_q = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} \qquad T\varepsilon_q T^t = T^t\varepsilon_q T = \varepsilon_q \det_q T. \tag{15}$$

The last relation, rewritten as $\sum_{mn}(\varepsilon_q^t)_{mn}T_{ni}T_{jm} = (\varepsilon_q^t)_{ji}$, allows us to introduce the trace operation for $K$-matrices which is invariant under the quantum group coaction (10) up to $\det_q T$ factors

$$\mathrm{tr}_q K \equiv \mathrm{tr}\, \varepsilon_q^t K = \mathrm{tr}_q TKT^t/\det_q T \qquad T \in F_q(\mathrm{GL}(2)). \tag{16}$$

Now we can prove that $c_1$ and $c_2$ defined by (9) lie in the centre of $\mathcal{A}$. Note that, by definition, $c_1 = \mathrm{tr}_q\, K$. Take $\mathrm{tr}_q$ of the relation (2) with respect to the first space in $\mathbb{C}^2 \otimes \mathbb{C}^2$. Then, using the fact that the $R$-matrix, as a solution to the YBE, is a representation of the algebra $F$ (4) having the quantum determinant $\det_q R = q$, and the identities (15), (16) one obtains from the LHS of (2) the equality

$$\overset{(1)}{\mathrm{tr}}_q (R \overset{1}{K} R^{t_1}) \overset{2}{K} = q c_1(K) \overset{2}{K}$$

and from the RHS of (2) respectively

$$\overset{2}{K}\, \overset{(1)}{\mathrm{tr}}_q (R^{t_1} \overset{1}{K} R) = \overset{2}{K} c_1(K) q$$

which establishes the commutativity of $c_1(K)$ with the generators $K$ of $\mathcal{A}$.

In order to obtain the expression for $c_2$ analogous to (13) multiply the RE (2) from the left by by the permutation operator $\mathcal{P}$

$$\hat{R} \overset{1}{K} R^{t_1} \overset{2}{K} = \overset{1}{K} R^{t_1} \overset{2}{K} \hat{R}$$

and then by the rank-one projector $P_-$. Taking into account (14) one gets

$$P_- \overset{1}{K} R^{t_1} \overset{2}{K} = P_- \overset{1}{K} R^{t_1} \overset{2}{K} P_- = c_2(K) P_- .$$

The proof of the commutativity of $c_2(K)$ with the generators $K$ of $\mathcal{A}$ uses the same fusion technique as the above proof for $\det_q T$. The fact that the quantum determinants of $R$ and $R^{t_1}$ are numbers

$$P_-^{(12)} R_{32} R_{31} = q P_-^{(12)} \qquad P_-^{(12)} R_{31}^{t_3} R_{32}^{t_3} = q P_-^{(12)}$$

again plays the crucial role.

Rewriting $q^2 c_2(K) \overset{3}{K}$ as

$$q^2 P_-^{(12)} \overset{1}{K} R^{t_1} \overset{2}{K} \overset{3}{K} = P_-^{(12)} \overset{1}{K} R_{12}^{t_1} \overset{2}{K} R_{31}^{t_3} R_{32}^{t_3} \overset{3}{K} R_{32} R_{31}$$

and then using the YBE for $R$ and $R^{t_1}$ and RE several times one finally arrives at

$$q^2 \overset{3}{K} P_-^{(12)} \overset{1}{K} R^{t_1} \overset{2}{K} = q^2 \overset{3}{K} c_2(K)$$

which proves the assertion.

Note that the central elements are transformed homogeneously under the quantum group coaction (10)

$$\varphi : c_1(K) \to \det_q T c_1(K) \qquad c_2(K) \to (\det_q T)^2 c_2(K). \tag{17}$$

The transformation law for $c_1(K)$ follows from (16). The corresponding formula for $c_2(K)$ follows from (15), (16) and the identity

$$\mathrm{tr}_q\, K \varepsilon_q K = (1+q^2) c_2(K) - q(c_1(K))^2.$$

Futhermore $K \varepsilon_q K = c_2 \varepsilon_q - q c_1 K$.

The formula (17) and the following inversion formula for the $K$-matrix

$$K^{-1} = \frac{1}{c_2} \begin{pmatrix} \delta & -\beta + \omega\gamma \\ -q^2\gamma & \alpha \end{pmatrix}$$

allow us to think of $c_2$ as the quantum deteminant of $K$.

We have no proof that for generic $q$ the centre of $\mathcal{A}$ is generated by $c_1$, $c_2$ though it is a highly plausible conjecture taking into account the analogous results for the quantum group $F$.

*3.4.* Let us now discuss the real forms of $\mathcal{A}$. It is well known that for $F_q(\mathrm{GL}(2))$ there are three real forms [5]. For $F_q(\mathrm{U}(2))$ the parameter $q$ is real $\bar{q} = q$ and the $*$-anti-involution is given by

$$\{a^*, b^*, c^*, d^*\} = \{d, -qc, -q^{-1}b, a\}.$$

For $F_q(\mathrm{U}(1,1))$ again $\bar{q} = q$ and

$$\{a^*, b^*, c^*, d^*\} = \{d, qc, q^{-1}b, a\}.$$

For $F_q(\mathrm{GL}(2,\mathbb{R}))$ the parameter $q$ is unitary $\bar{q} = 1/q$ and

$$\{a^*, b^*, c^*, d^*\} = \{a, b, c, d\}.$$

The corresponding real forms of $\mathcal{A}$ are

$$\begin{aligned} &F_q(\mathrm{U}(2)): && \{\alpha^*, \beta^*, \gamma^*, \delta^*\} = \{q\delta, -\beta, -\gamma, q^{-1}\alpha\} \\ &F_q(\mathrm{U}(1,1)): && \{\alpha^*, \beta^*, \gamma^*, \delta^*\} = \{q\delta, \beta, \gamma, q^{-1}\alpha\} \\ &F_q(\mathrm{GL}(2,\mathbb{R}): && \{\alpha^*, \beta^*, \gamma^*, \delta^*\} = \{\alpha, \gamma + qc_1, q\gamma, \delta\}. \end{aligned}$$

It is easy to verify that these real forms of $\mathcal{A}$ are consistent with those of $F$, that is $\varphi(K^*_{ij}) = \varphi(K_{ij})^*$.

*3.5.* The invariance of the ideal $c_1 = 0$ under the coaction $\varphi$ follows immediately from (17). The resulting quotient algebra is generated by three generators $\alpha$, $\gamma$, $\delta$ ($\beta = q\gamma$) and the relations

$$\alpha\gamma = q^2\gamma\alpha \qquad \gamma\delta = q^2\delta\gamma \qquad [\alpha, \delta] = q(q^2 - q^{-2})\gamma^2.$$

The algebra is isomorphic to the quantum homogeneous space for $F_q(\mathrm{SL}(2))$. It is also isomorphic to the subalgebra of $F$ generated by the elements of the matrix $TT^t$ [5] (see [18] for the $F_q(\mathrm{GL}(n))$ case).

## 4. Representations

The representation theory for the algebra $\mathcal{A}$ is a topic deserving special investigation. We restrict ourselves to a few remarks.

There are two one-dimensional representations of $\mathcal{A}$

$$K^{(0)} = \varepsilon_q = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} \qquad K^{(1)} = \begin{pmatrix} \lambda & \mu \\ 0 & \nu \end{pmatrix}. \tag{18}$$

As shown in the next section, under the additional condition of $\gamma$ being invertible the quotient algebra $\mathcal{A}/(c_2 = 1)$ is isomorphic to the quantum algebra $\mathcal{U}_q(\mathrm{sl}(2))$. Hence, the irreducible representations of $\mathcal{U}_q(\mathrm{sl}(2))$ are translated into those of $\mathcal{A}$.

Another way to construct representations of $\mathcal{A}$ is to use the coaction $\varphi$ (10) of $F$. Any pair of representations $\pi$ of $\mathcal{A}$ and $\rho$ of $F$

$$\pi : \mathcal{A} \to \mathrm{End}(V) \qquad \rho : F \to \mathrm{End}(W)$$

gives rise to another representation of $\mathcal{A}$

$$(\rho \otimes \pi) \circ \varphi : \mathcal{A} \to \mathrm{End}(W \otimes V).$$

It is an open question whether this construction generates a new irreducible representation of $\mathcal{A}$ provided $\pi$ and $\rho$ are irreducible. It is also unknown if any irreducible representation of $\mathcal{A}$ can be decomposed into $\pi$ and $\rho$.

Representations of comodule algebras related to the reflection equations with $R$-matrices corresponding to other quantum (super)groups such as $F_q(\mathrm{GL}(m|n))$ were studied in [19].

## 5. Variants of RE and related algebras

*5.1.* Consider the algebra $\mathcal{X}$ defined by the generators $x$, $y$, $u$, $v$ and by the relations

$$xy = qyx \qquad xu = qux \qquad xv = vx + \omega uy$$
$$uv = qvu \qquad yu = uy \qquad yv = qvy.$$

Introducing the column $X$ and row $Y$

$$X = \begin{pmatrix} x \\ y \end{pmatrix} \qquad Y = (u \quad v)$$

one can represent the above commutation relations as the exchange algebras [7, 8, 11]

$$R \overset{1}{X} \overset{2}{X} = q \overset{2}{X} \overset{1}{X} \qquad \overset{2}{Y} \overset{1}{Y} R = q \overset{1}{Y} \overset{2}{Y} \tag{19}$$

$$\overset{1}{Y} R^{t_1} \overset{2}{X} = \overset{2}{X} \overset{1}{Y} \qquad \overset{2}{Y} R^{t_1} \overset{1}{X} = \overset{1}{X} \overset{2}{Y}. \tag{20}$$

The algebra $\mathcal{X}$ has a central element $\zeta = Y \varepsilon_q X = uy - qvx$. The subalgebras of $\mathcal{X}$ spanned by $X$ or $Y$ are isomorphic to the 'quantum plane' [13].

By virtue of the relations (19), (20) the matrix

$$K = XY = \begin{pmatrix} xu & xv \\ yu & yv \end{pmatrix}$$

satisfies the RE (2), the above formula thus giving a homomorphism $\mathcal{A} \to \mathcal{X}$. Note that for the realization of $\mathcal{A}$ obtained one has $c_2(K) = 0$ and $K \varepsilon_q K = \zeta K = -qc_1(K)K$.

*5.2.* In the papers [6, 7] another version of the RE was obtained

$$R\overset{1}{M}R^{-1}\overset{2}{M} = \overset{2}{M}\tilde{R}^{-1}\overset{1}{M}\tilde{R} \qquad \tilde{R} \equiv \mathcal{P}R\mathcal{P} \qquad M = \begin{pmatrix} \xi & \eta \\ \theta & \tau \end{pmatrix}. \tag{21}$$

The corresponding algebra $\mathcal{B}$ is an $F$-comodule algebra with the coaction

$$\psi : \mathcal{B} \to F \otimes \mathcal{B} \qquad \psi(M) = TMT^{-1}.$$

In the case of $F = F_q(\mathrm{GL}(2))$ the $F$-comodule algebra $\mathcal{B}$ is isomorphic to $\mathcal{A}$ because of the relation (15). The isomorphism is given by the formula

$$K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} -q\eta & \xi \\ -q\tau & \theta \end{pmatrix} = M\varepsilon_q$$

which implies the relations between the central elements

$$c_1(K) = \beta - q\gamma = \xi + q^2\tau \equiv z_1(M)$$
$$c_2(K) = \alpha\delta - q^2\beta\gamma = q^3(\xi\tau - q^{-2}\eta\theta) \equiv q^3 z_2(M).$$

Under this isomorphism the $q$-trace (16) maps into $\operatorname{tr} DM$, $D = \operatorname{diag}(q^{-1}, q)$ introduced in [5], see also [10].

The algebra $\mathcal{B}$ is connected closely to the quantum algebra $\mathcal{U}_q(\mathrm{sl}(2))$ [6, 7]. Namely, the matrix $M$ can be realized in terms of $L^{\pm}$ (12)

$$M = s(L^+)L^- \tag{22}$$

where $s$ is the antipode in $\mathcal{U}_q(\mathrm{sl}(2))$: $s(H) = -H$, $s(X_\pm) = -q^{\mp 1}X_\pm$.

The formula (22) describes the algebra homomorphism $\chi : \mathcal{B} \to \mathcal{U}_q(\mathrm{sl}(2))$

$$\chi(M) = \begin{pmatrix} q^{-2H} + q\omega^2 X_- X_+ & -\omega q X_- q^H \\ -\omega q^H X_+ & q^{2H} \end{pmatrix}$$

for which $z_1(M)$ is proportional to the well known Casimir operator for $\mathcal{U}_q(\mathrm{sl}(2))$ and $z_2(M) = 1$ [10].

It is easy to see that the inversion of the homomorphism $\chi$ needs invertibility of the element $\tau$ (which is not the case for the one-dimensional representation $M = K^{(1)}\varepsilon_q$, see (18)). If the element $\tau$ is supposed to be invertible one can introduce in $\mathcal{B}$ a coproduct induced from $\mathcal{U}_q(\mathrm{sl}(2))$ [7].

*5.3.* We conclude this section with describing the classical limit of $\mathcal{A}$. Let $q = \mathrm{e}^h$. As $h \to 0$ or $q \to 1$ the algebra $\mathcal{A}$ becomes commutative, its commutator giving rise to the Poisson bracket $[\, ,\,] = -h\{\, ,\,\}$. The commutation relations (2) are transformed into the Poisson brackets relations

$$\{\overset{1}{K}, \overset{2}{K}\} = [r, \overset{1}{K}\overset{2}{K}] + \overset{1}{K}r^{t_1}\overset{2}{K} - \overset{2}{K}r^{t_1}\overset{1}{K} \tag{23}$$

where $r$ is the classical $R$-matrix $R = 1 + hr + \mathrm{O}(h^2)$

$$r = \begin{pmatrix} 1 & & & \\ & & & \\ & & 2 & \\ & & & 1 \end{pmatrix} \qquad r^{t_1} = \begin{pmatrix} 1 & & & 2 \\ & & & \\ & & & \\ & & & 1 \end{pmatrix}.$$

Fixing values of the central functions $c_1 = \beta - \gamma$ and $c_2 = \det K = \alpha\delta - \beta\gamma$ one obtains the foliation of the 4-dimensional space spanned by $\alpha\beta\gamma\delta$ into 2-dimensional manifolds on which the Poisson bracket (23) is non-degenerate.

## 6. Generalizations

*6.1.* Let us discuss now the fusion procedure for the RE. It does not differ much from the case of the YBE [5, 16]. The peculiarities of the RE case are well seen in the simplest spin-1 case. The $R$-matrix in (4) $R \in \mathbb{C}^2 \otimes \mathbb{C}^2$ intertwines two spin-1/2 corepresentations of $F_q(\mathrm{GL}(2))$. The $R$-matrix in $\mathbb{C}^2 \otimes \mathbb{C}^3$ is

$$R_{1(2)} = P_+ R_{12'} R_{12} = P_+ R_{12'} R_{12} P_+$$

where $P_+$ is the rank-3 projector (14) from $\mathbb{C}^2 \otimes \mathbb{C}^2$ onto $\mathbb{C}^3$ and the corresponding spaces are labelled respectively by the indices 2, $2'$ and $(2)$. We shall also use

$$R^{t_1}_{1(2)} = P_+ R^{t_1}_{12} R^{t_1}_{12'} P_+ .$$

The $3 \times 3$ matrix $J$

$$J = P_+ \overset{2}{K} R^{t_1}_{22'} \overset{2'}{K} P_+$$

satisfies the RE in $\mathbb{C}^2 \otimes \mathbb{C}^3$

$$R_{1(2)} \overset{1}{K} R^{t_1}_{1(2)} \overset{(2)}{J} = \overset{(2)}{J} R^{t_1}_{1(2)} \overset{1}{K} R_{1(2)}$$

and also the RE in $\mathbb{C}^3 \otimes \mathbb{C}^3$

$$R_{(1)(2)} \overset{(1)}{J} R^{t_1}_{(1)(2)} \overset{(2)}{J} = \overset{(2)}{J} R^{t_1}_{(1)(2)} \overset{(1)}{J} R_{(1)(2)} .$$

The entries of the matrix $J$ generate a subalgebra $\mathcal{A}'$ of $\mathcal{A}$ which is also an $F$-co-ideal that is $\varphi : \mathcal{A}' \to F \otimes \mathcal{A}'$. The analogous fusion procedure for the spectral parameter dependent RE (1) was developed in [20].

There is no doubt that, being properly generalized, the above described fusion procedure should be able to produce solutions to the RE in matrices of any dimension corresponding to higher finite-dimensional irreducible corepresentations of $F_q(\mathrm{GL}(2))$. At least for the one-dimensional representations of $\mathcal{A}$ (18) the expression for the 'universal $K$-matrix' can be written explicitly (see [21] for $K^{(1)}$ and [12, 19] for $K^{(0)}$).

*6.2.* Throughout this paper we considered the algebra $\mathcal{A}$ defined by the relations (2) corresponding to the $R$-matrix (5) of the quantum group $F_q(\mathrm{GL}(2))$. However, the proof of the comodule algebra property does not use anything besides the commutation relations (4) and the symmetries (11) of the $R$-matrix. Therefore, to any $R$-matrix satisfying (11) and the related quantum group $F_R$ defined by relations (4) there corresponds some $F_R$-comodule algebra $\mathcal{A}_R$ defined by relations (2).

The conditions (11), however, are not essential and can be disposed of. Let us extend the algebra $F_R$ introducing another matrix of generators $S$ in addition to $T$ and imposing the commutation relations

$$\begin{aligned} R^{(1)} \overset{1}{T} \overset{2}{T} &= \overset{2}{T} \overset{1}{T} R^{(1)} \qquad & \overset{1}{S} R^{(2)} \overset{2}{T} &= \overset{2}{T} R^{(2)} \overset{1}{S} \\ \overset{1}{T} R^{(3)} \overset{2}{S} &= \overset{2}{S} R^{(3)} \overset{1}{T} \qquad & R^{(4)} \overset{1}{S} \overset{2}{S} &= \overset{2}{S} \overset{1}{S} R^{(4)} \end{aligned} \tag{24}$$

parametrized by four $R$-matrices. It is easy to verify that if $K$ obeys the relation

$$R^{(1)}\overset{1}{K}R^{(2)}\overset{2}{K} = \overset{2}{K}R^{(3)}\overset{1}{K}R^{(4)} \tag{25}$$

then the matrix $TKS$ obeys the same relation provided matrix elements of $K$ commute with those of $T$ and $S$.

Suppose now that $S = \sigma(T)$ where $\sigma$ is an anti-automorphism of matrix algebra that is $\sigma(xy) = \sigma(y)\sigma(x)$ for any number matrices $x$, $y$. Then all the $R$-matrices in (25) can be expressed in terms of the matrix $R$ defining the quantum group (4). The resulting algebra is an $F_R$-comodule algebra with the coaction $K \to TK\sigma(T)$.

Note that there exists a remarkable ambiguity in the choice of the $R$-matrices. Namely, one can make the substitution $R \to \mathcal{P}R^{-1}\mathcal{P}$ independently in any of the matrices $R^{(i)}$ which does not affect the commutation relations (24), cf. the remark immediately after formula (5). Consequentely, there exist at least $2^4 = 16$ *a priori* non-equivalent algebras sharing the same comodule structure. The interrelations of these algebras will be described in a separate paper.

In the case of $\sigma(T) = T^t$ the $R$-matrices are

$$\begin{aligned}
R^{(1)} &= R && \text{or} \quad \widetilde{R}^{-1} \\
R^{(2)} &= R^{t_1} && \text{or} \quad (\widetilde{R}^{-1})^{t_1} \\
R^{(3)} &= (\widetilde{R})^{t_2} && \text{or} \quad (R^{-1})^{t_2} \\
R^{(4)} &= \widetilde{R}^{t} && \text{or} \quad (R^{-1})^{t}
\end{aligned}$$

where, as in (21), the notation $\widetilde{R} \equiv \mathcal{P}R\mathcal{P}$ is used. Choosing the first option for every $R^{(i)}$ one obtains from (25) the equation [19]

$$R\overset{1}{K}R^{t_1}\overset{2}{K} = \overset{2}{K}\widetilde{R}^{t_2}\overset{1}{K}\widetilde{R}^{t} \tag{26}$$

coinciding with the RE (2) if $R$ has the symmetries (11).

Another example is provided by $\sigma(T) = T^{-1}$. In this case

$$\begin{aligned}
R^{(1)} &= R && \text{or} \quad \widetilde{R}^{-1} \\
R^{(2)} &= R^{-1} && \text{or} \quad \widetilde{R} \\
R^{(3)} &= \widetilde{R}^{-1} && \text{or} \quad R \\
R^{(4)} &= \widetilde{R} && \text{or} \quad R^{-1}.
\end{aligned}$$

The choice of the first option for every $R^{(i)}$ leads to the algebra $\mathcal{B}$ described in section 5.2.

*6.3.* We conclude our discussion of the RE with a remark concerning the very form of the equation. Note that formula (10) for the coaction of $F$ is equivalent to

$$\varphi : K^{i_1 i_2} \to T^{i_1}_{j_1} T^{i_2}_{j_2} K^{j_1 j_2} \tag{27}$$

where the indices are shown explicitly. Here we use the commutativity of $T$ and $K$ and assume summation over repeated indices.

Formula (27) suggests that $K$ can be thought of as a bivector (contravariant tensor of the second order). In a compact form one can write (27) as

$$\varphi : K_{12} \to \overset{1}{T}\overset{2}{T} \cdot K_{12} \tag{28}$$

(the notation is self-evident).

Analogously, equation (26) can be rewritten as

$$R^{i_1 i_2}_{j_1 j_2} K^{j_1 j_1'} R^{i_1' j_2}_{j_1' k_2} K^{k_2 i_2'} = K^{i_2 j_2'} R^{k_2' i_1}_{j_2' j_1} K^{j_1 j_1'} R^{i_2' i_1'}_{k_2' j_1'}$$

or, using the same notation as in (28)

$$R_{12} R_{1'2} \cdot K_{11'} K_{22'} = \widetilde{R}_{1'2'} \widetilde{R}_{12'} \cdot K_{22'} K_{11'} . \tag{29}$$

Moving two $R$-matrices from the right-hand side of (29) to the left-hand side one obtains

$$\widetilde{R}^{-1}_{12'} \widetilde{R}^{-1}_{1'2'} R_{12} R_{1'2} \cdot K_{11'} K_{22'} = K_{22'} K_{11'} . \tag{30}$$

Combining the two indices in $K^{i_1 i_2}$ into one composite index and the four $R$-matrices in (30) into one composite $R$-matrix we observe that equation (30) describes a sort of exchange algebra (19).

The case $\sigma(T) = T^{-1}$ leads in the same way to a mixed tensor having one upper and one lower index.

This observation opens a road to a far-going generalization of the algebra $\mathcal{A}$. Acting in the same spirit one can introduce algebras corresponding to the quantum tensors with any number of upper and lower indices. The papers [13, 22] where the quantum multilinear algebras are studied might be of relevance in performing this program. We hope to touch on this subject in a separate article.

## Acknowledgments

We are grateful to our colleagues at YITP and RIMS of Kyoto University for warm hospitality and useful discussions.

## References

[1] Sklyanin E 1988 *J. Phys. A: Math. Gen.* **21** 2375
[2] Cherednik I 1984 *Theor. Math. Phys.* **61** 55
[3] Reshetikhin N and Semenov-Tian-Shansky M 1990 *Lett. Math. Phys.* **19** 133
[4] Kulish P and Sklyanin E 1991 *J. Phys. A: Math. Gen.* **24** L435
[5] Faddeev L, Reshetikhin N and Takhtajan L 1989 *Algebra i Analiz* **1** 178
Takhtajan L 1990 *Lecture Notes in Physics* **370** 3
[6] Moore G and Reshetikhin N 1989 *Nucl. Phys.* B **328** 557
[7] Alekseev A, Faddeev L and Semenov-Tian-Shansky M 1992 *Lecture Notes in Mathematics* **1510** 148
Alekseev A and Faddeev L 1991 *Commun. Math. Phys.* **141** 413
[8] Babelon O 1991 *Commun. Math. Phys.* **139** 619
[9] Freidel L and Maillet J 1991 *Phys. Lett.* **262B** 278; 1991 *Phys. Lett.* **263B** 403

[10] Zumino B 1991 Introduction to the differential geometry of quantum groups *Preprint* UCB-PTH-62/91
[11] Kulish P 1991 *Phys. Lett.* **161A** 50
[12] Kulish P 1991 Quantum groups and quantum algebras as symmetries of dynamical systems *Preprint* YITP/K-959
[13] Manin Yu 1991 *Topics in Non-commutative Geometry* (Princeton, NJ: Princeton University Press)
[14] Rosso M 1988 *Commun. Math. Phys.* **117** 581
[15] Majid S 1990 *Int. J. Mod. Phys.* A **5** 1; 1991 *J. Math. Phys.* **32** 3246
[16] Kulish P, Reshetikhin N and Sklyanin E 1981 *Lett. Math. Phys.* **5** 393
[17] Jimbo M 1986 *Lett. Math. Phys.* **11** 247
[18] Jing N and Yamada H 1990 Polynômes zonaux pour le groupe linéaire général quantique *Preprint* IAS
Noumi M 1992 Macdonald's symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces *Preprint* UT-Komaba
[19] Kulish P, Sasaki R and Schwiebert C 1992 Constant solutions of reflection equations *Preprint* YITP/U-92-07
[20] Mezincescu L and Nepomechie R 1991 Fusion procedure for open chains *Preprint* CERN-TH. 6152/91
[21] Cremmer E and Gervais J-L 1992 *Commun. Math. Phys.* **144** 279
[22] Hashimoto M and Hayashi T 1991 Quantum multilinear algebra *Preprint* Nagoya University
Gurevich D and Rubtsov V 1992 *Lecture Notes in Mathematics* **1510** 47